A W_2^1-theory of Stochastic Partial Differential Systems of Divergence type on C^1 domains

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Abstract

In this paper we study the stochastic partial differential systems of divergence type with C^1 space domains in $\mathbb{R}^d$. Existence and uniqueness results are obtained in terms of Sobolev spaces with weights so that we allow the derivatives of the solution to blow up near the boundary. The coefficients of the systems are only measurable and are allowed to blow up near the boundary.

Keywords: stochastic parabolic partial differential systems, divergence type, weighted Sobolev spaces.

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1 Introduction

In this article we are dealing with W_2^1-theory of the stochastic partial differential systems (SPDSs) of d_1 equations of divergent type:

$$\begin{aligned} du^k &= (D_i(a^{ij}_{kr}u^r_{x^j} + \bar{b}^i_{kr}u^r + \bar{f}^{ik}) + b^i_{kr}u^r_{x^i} + c_{kr}u^r + f^k)dt \\ &\quad +(\sigma^i_{kr,m}u^r_{x^i} + \nu_{kr,m}u^r + g^k_m)dw^m_t, \qquad t>0 \\ u^k(0) &= u^k_0 \end{aligned} \tag{1.1}$$

with $x \in \mathbb{R}^d$, $\mathbb{R}^d_+$ or $\mathcal{O}$, a bounded C^1 domain. Here, $\{w^m_t : m = 1,2,\ldots\}$ is a countable set of independent one-dimensional Brownian motions defined on a probability space $(\Omega, \mathcal{F}, P)$. Indices i and j run from 1 to d while $k,\ j = 1,2,\cdots,d_1$ and $m = 1,2,\cdots$. To make expressions simple, we are using the summation convention on i,j,r,m. The coefficients $a^{ij}_{kr}, \bar{b}^i_{kr}, b^i_{kr}, c_{kr}, \sigma^i_{kr,m}$ and $\nu_{kr,m}$ are measurable functions depending on $\omega \in \Omega, t, x$. Detailed formulation of (1.1) follows in the subsequent sections.

Demand for a general theory of stochastic partial differential systems(SPDSs) arises when we model the interactions among unknowns in a natural phenomenon with random behavior. For example, the motion of a random string can be modeled by means of SPDSs(see [14] and [1]).

We note that, if $d_1 = 1$, then the system (1.1) becomes a single stochastic partial differential equation (SPDE) of divergence type. In this case L_2-theory on $\mathbb{R}^d$ was developed long ago and an

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account of it can be found, for instance, in [15]. Also, L_p-theory($p \geq 2$) of such single equations with C^1 space domains can be found in [3], [5] and [16] in which weighted Sobolev spaces are used to allow derivatives of the solutions to blow up near the boundary. For comparison with L_p-theory of SPDEs of non-divergence type, we refer to [4], [7], [12], [10] and references therein.

The main goal of this article is to extend the results [15], [3], [5], [16] for single equations to the case of systems under no smoothness assumptions on the coefficients. We prove the uniqueness and existence results of system (1.1) in weighted Sobolev spaces so that we allow the derivatives of the solutions to blow up near the boundary. The coefficients of the system are only measurable and are allowed to blow up near the boundary (See (4.6)).

We declare that W^1_p-theory, a desirable further result beyond W^1_2-theory, is not successful yet even under the assumption that the coefficients a^{ij}_{kr} and σ^i_{kr} are constants. This is due to the difficulty caused by considering SPDSs instead of SPDEs. For L_p-theory, $p > 2$, one must overcome tremendous mathematical difficulties rising in the general settings; one of the main difficulties in the case $p > 2$ is that the arguments we are using in the proof of Lemma 3.3 below are not working since in this case we get some extra terms which we simply can not control.

The organization of the article is as follows. Section 2 handles the Cauchy problem. In section 3 and section 4 we develop our theory of the system defined on $\mathbb{R}^d_+$ and bounded domain $\mathcal{O}$, respectively.

As usual, $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, ..., x^d)$, $B_r(x) = \{y \in \mathbb{R}^d : |x-y| < r\}$, $B_r = B_r(0)$ and $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^1 > 0\}$. For $i = 1, ..., d$, multi-indices $\alpha = (\alpha_1, ..., \alpha_d)$, $\alpha_i \in \{0, 1, 2, ...\}$, and functions $u(x)$ we set

$$u_{x^i} = \frac{\partial u}{\partial x^i} = D_i u, \quad D^\alpha u = D_1^{\alpha_1} \cdot ... \cdot D_d^{\alpha_d} u, \quad |\alpha| = \alpha_1 + ... + \alpha_d.$$

If we write $c = c(\cdots)$, this means that the constant c depends only on what are in parenthesis.

2 The systems on $\mathbb{R}^d$

In this section we develop some solvability results of linear systems defined on space domain $\mathbb{R}^d$. These results will be used later for systems defined on $\mathbb{R}^d_+$ or a bounded C^1 domain $\mathcal{O}$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\{\mathcal{F}_t\}$ be a filtration such that $\mathcal{F}_0$ contains all P-null sets of Ω; the probability space $(\Omega, \mathcal{F}, P)$ is rich so that we define independent one-dimensional $\{\mathcal{F}_t\}$-adapted Wiener processes $\{w^m_t\}_{m=1}^\infty$ on it. We let $\mathcal{P}$ denote the predictable σ-algebra on $\Omega \times (0, \infty)$.

The space $C^\infty_0 = C^\infty_0(\mathbb{R}^d; \mathbb{R}^{d_1})$ denotes the set of all $\mathbb{R}^{d_1}$-valued infinitely differentiable functions with compact support in $\mathbb{R}^d$. By $\mathcal{D}$ we mean the space of $\mathbb{R}^{d_1}$-valued distributions on C^∞_0; precisely, for $u \in \mathcal{D}$ and $\phi \in C^\infty_0$ we define $(u, \phi) \in \mathbb{R}^{d_1}$ with components $(u, \phi)^k = (u^k, \phi^k)$, $k = 1, \ldots, d_1$. Each u^k is a usual $\mathbb{R}$-valued distribution defined on $C^\infty(\mathbb{R}^d; \mathbb{R})$. We let $L_p = L_p(\mathbb{R}^d; \mathbb{R}^{d_1})$ be the space of all $\mathbb{R}^{d_1}$-valued functions $u = (u^1, \ldots, u^{d_1})$ satisfying

$$\|u\|^p_{L_p} := \sum_{k=1}^{d_1} \|u^k\|^p_{L_p} < \infty.$$

For $p \in [2,\infty)$ and $\gamma \in (-\infty,\infty)$ we define the space of Bessel potential $H_p^\gamma = H_p^\gamma(\mathbb{R}^d;\mathbb{R}^{d_1})$ as the space of all distributions u such that $(1-\Delta)^{n/2}u \in L_p$, where

$$((1-\Delta)^{\gamma/2}u)^k := (1-\Delta)^{\gamma/2}u^k := \mathcal{F}^{-1}[(1+|\xi|^2)^{\gamma/2}\mathcal{F}(u^k)(\xi)].$$

Here, $\mathcal{F}$ is the Fourier transform. Define

$$\|u\|_{H_p^\gamma} := \|(1-\Delta)^{\gamma/2}u\|_{L_p}.$$

Then, H_p^γ is a Banach space with the given norm and C_0^∞ is dense in H_p^γ. Note that H_p^γ are usual Sobolev spaces for $\gamma = 0,1,2,\ldots$. It is well known that the first order differentiation operators, $\partial_i : H_p^\gamma(\mathbb{R}^d;\mathbb{R}) \to H_p^{\gamma-1}(\mathbb{R}^d;\mathbb{R})$ given by $u \to u_{x^i}$ $(i=1,2,\ldots,d)$, are bounded. On the other hand, for $u \in H_p^\gamma(\mathbb{R}^d;\mathbb{R})$, if $\mathrm{supp}\,(u) \subset (a,b)\times\mathbb{R}^{d-1}$ with $-\infty < a < b < \infty$, we have

$$\|u\|_{H_p^\gamma(\mathbb{R}^d;\mathbb{R})} \le c(d,\gamma,a,b)\|u_x\|_{H_p^{\gamma-1}(\mathbb{R}^d;\mathbb{R})} \tag{2.1}$$

(see, for instance, Remark 1.13 in [11]). Let ℓ_2 be the set of all real-valued sequences $e=(e_1,e_2,\ldots)$ with the inner product $(e,f)_{\ell_2} = \sum_{m=1}^\infty e_m f_m$ and the norm $|e|_{\ell_2} := (e,e)_{\ell_2}^{1/2}$. For $g=(g^1,g^2,\cdots,g^{d_1})$, where g^k are ℓ_2-valued functions, we define

$$\|g\|_{H_p^\gamma(\ell_2)}^p := \sum_{k=1}^{d_1} \||(1-\Delta)^{\gamma/2}g^k|_{\ell_2}\|_{L_p}^p.$$

Using the spaces mentioned above, for a fixed time T, we define the stochastic Banach spaces

$$\mathbb{H}_p^\gamma(T) := L_p(\Omega\times(0,T],\mathcal{P},H_p^\gamma), \quad \mathbb{H}_p^\gamma(T,\ell_2) := L_p(\Omega\times(0,T],\mathcal{P},H_p^\gamma(\ell_2)),$$

$$\mathbb{L}_p(T) := \mathbb{H}_p^0(T), \quad \mathbb{L}_p(T,\ell_2) = \mathbb{H}_p^0(T,\ell_2)$$

with norms given by

$$\|u\|_{\mathbb{H}_p^\gamma(T)}^p = \mathbb{E}\int_0^T \|u(t)\|_{H_p^\gamma}^p dt, \quad \|g\|_{\mathbb{H}_p^\gamma(T,\ell_2)}^p = \mathbb{E}\int_0^T \|u(t)\|_{H_p^\gamma(\ell_2)}^p dt.$$

Lastly, we set $U_p^\gamma = L_p(\Omega,\mathcal{F}_0,H_p^{\gamma-2/p})$.

Definition 2.1. For a $\mathcal{D}$-valued function $u \in \mathbb{H}_p^{\gamma+2}(T)$, we write $u \in \mathcal{H}_p^{\gamma+2}(T)$ if $u(0,\cdot) \in U_p^{\gamma+2}$ and there exist $f \in \mathbb{H}_p^\gamma(T)$, $g \in \mathbb{H}_p^{\gamma+1}(T,\ell_2)$ such that

$$du = f\,dt + g^m dw_t^m, \quad t \le T$$

in the sense of distributions, that is, for any $\phi \in C_0^\infty$ and $k=1,2,\cdots,d_1$, the equality

$$(u^k(t,\cdot),\phi) = (u^k(0,\cdot),\phi) + \int_0^t (f^k(s,\cdot),\phi)ds + \sum_{m=1}^\infty \int_0^t (g_m^k(s,\cdot),\phi)dw_s^m \tag{2.2}$$

holds (a.s.) for all $t \le T$. The norm in $\mathcal{H}_p^{\gamma+2}(T)$ is defined by

$$\|u\|_{\mathcal{H}_p^{\gamma+2}(T)} = \|u\|_{\mathbb{H}_p^{\gamma+2}(T)} + \|f\|_{\mathbb{H}_p^\gamma(T)} + \|g\|_{\mathbb{H}_p^{\gamma+1}(T,\ell_2)} + \|u(0)\|_{U_p^{\gamma+2}}.$$

Remark 2.2. Note that since the coefficients in system (1.1) are only measurable, the space $\mathcal{H}_p^{\gamma+2}(T)$ is not appropriate for system (1.1) unless $\gamma = -1$.

We set $A^{ij} = (a^{ij}_{kr})$, $\Sigma^i = (\sigma^i_{kr})$ and $\mathcal{A}^{ij} = (\alpha^{ij}_{kr})$, where

$$\alpha^{ij}_{kr} = \frac{1}{2}\sum_{l=1}^{d}(\sigma^i_{lk}, \sigma^j_{lr})_{\ell_2}, \quad \sigma^i_{kr} = (\sigma^i_{kr,1}, \sigma^i_{kr,2}, \cdots).$$

Also, we set $\bar{B}^i = (\bar{b}^i_{kr}), B^i = (b^i_{kr}), C = (c_{kr}), \mathcal{N} = (\nu_{kr})$, where $\nu_{kr} := (\nu_{kr,1}, \nu_{kr,2}, \ldots)$.

For any $d_1 \times d_1$ matrix $M = (m_{kr})$ we let

$$|M| := \sqrt{\sum_{k,r}(m_{kr})^2}\ ; \quad |M| := \sqrt{\sum_{k,r}|m_{kr}|^2_{\ell_2}},$$

where the latter is the case that the elements are in ℓ_2.

Throughout the article we assume the following.

Assumption 2.3. (i) The coefficients $a^{ij}_{kr}, \bar{b}^i_{kr}, b^i_{kr}, c_{kr}, \sigma^i_{kr,m}$ and $\nu_{kr,m}$ are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable, where $\mathcal{B}(\mathbb{R}^d)$ denotes Borel σ-field in $\mathbb{R}^d$.

(ii) There exist finite constants $\delta,\ K^j (j = 1, \ldots, d),\ L > 0$ so that

$$\delta|\xi|^2 \leq \xi^*_i \left(A^{ij} - \mathcal{A}^{ij}\right)\xi_j \tag{2.3}$$

holds for any $\omega \in \Omega,\ t > 0$, where ξ is any (real) $d_1 \times d$ matrix, ξ_i is the ith column of ξ; again the summations on i, j are understood. Moreover, we assume that for any $\omega,\ t > 0,\ x \in \mathbb{R}^d$, $i, j = 1, \ldots, d$,

$$\left|A^{1j}(\omega, t, x)\right| \leq K^j, \quad \left|A^{ij}(\omega, t, x)\right| \leq L\ (i \neq 1), \quad |\mathcal{A}^{ij}(\omega, t, x)| \leq L. \tag{2.4}$$

Our main theorem in this section is the following.

Theorem 2.4. *Assume that there is a constant $N_0 \in (1, \infty)$ such that for any $\omega,\ t > 0,\ x \in \mathbb{R}^d$, $i = 1, \ldots, d$,*

$$|\bar{B}^i|,\ |B^i|,\ |C|,\ |\mathcal{N}| < N_0. \tag{2.5}$$

Then for any $\bar{f}^i \in \mathbb{L}_2(T)$ $(i = 1, \ldots, d)$, $f \in \mathbb{H}^{-1}_2(T)$, $g \in \mathbb{L}_2(T, \ell_2)$, and $u_0 \in U^1_2$, system (1.1) has a unique solution $u \in \mathcal{H}^1_2(T)$, and for this solution we have

$$\|u_x\|_{\mathbb{L}_2(T)} \leq c(\|u\|_{\mathbb{L}_2(T)} + \sum_i \|\bar{f}^i\|_{\mathbb{L}_2(T)} + \|f\|_{\mathbb{H}^{-1}_2(T)} + \|g\|_{\mathbb{L}_2(T,\ell_2)} + \|u_0\|_{U^1_2}), \tag{2.6}$$

$$\|u\|_{\mathbb{H}^1_2(T)} \leq ce^{cT}(\sum_i \|\bar{f}^i\|_{\mathbb{L}_2(T)} + \|f\|_{\mathbb{H}^{-1}_2(T)} + \|g\|_{\mathbb{L}_2(T,\ell_2)} + \|u_0\|_{U^1_2}), \tag{2.7}$$

where $c = c(d, d_1, \delta, K, L, N_0)$.

Proof. 1. We note that f^k can be expressed as $f^k = F^{0k} + \mathrm{div}(F^{1k}, F^{2k}, \ldots, F^{dk})$, where $F^{0k} \in \mathbb{H}^1_2(T)$, $F^{ik} \in \mathbb{H}^{\gamma}_2(T)$ with the estimate $\|F^{0k}\|_{\mathbb{H}^1_2(T)} + \sum_{i=1}^d \|F^{ik}\|_{\mathbb{L}_2(T)} \leq c(d, d_1)\|f^k\|_{\mathbb{H}^{-1}_2(T)}$; this

follows from the observation $f^k=(1-\Delta)(1-\Delta)^{-1}f^k=(1-\Delta)^{-1}f^k+\mathrm{div}(-\nabla((1-\Delta)^{-1}f^k))$ (see, p.197 of [11]). Hence, we may assume that $f\in\mathbb{H}^1_2(T)$ and show (2.6) and (2.7) with $\|f\|_{\mathbb{H}^1_2(T)}$ in place of $\|f\|_{\mathbb{H}^{-1}_2(T)}$.

2. By Theorem 4.10 and Theorem 5.1 in [10], for each k the equation

$$du^k=\left(D_i(\delta\cdot\delta_{ij}\delta_{kr}u^r_{x^ix^j}+\bar{f}^{ik})+f^k\right)dt+g^k_m dw^m_t,$$

or equivalently,

$$du^k=(\delta\Delta u^k+\bar{f}^{ik}_{x^i}+f^k)dt+g^k_m dw^m_t,\quad u^k(0)=u^k_0,$$

has a solution u^k and we have $u:=(u^1,u^2,\cdots,u^{d_1})^*$ as the unique solution of

$$du=(\delta\Delta u+\bar{f}^i_{x^i}+f)dt+g_m dw^m_t,\quad u(0)=u_0,$$

in $\mathcal{H}^1_2(T)$ with estimates (2.6) and (2.7). For $\lambda\in[0,1]$ we define

$$\begin{aligned}E^{ij}_\lambda=(e^{ij}_{kr,\lambda}) &:= (1-\lambda)\left(A^{ij}-\mathcal{A}^{ij}\right)+\lambda\delta\cdot\delta_{ij}I\\ &= \left((1-\lambda)A^{ij}+\lambda\delta\cdot\delta_{ij}I\right)-(1-\lambda)\mathcal{A}^{ij}=A^{ij}_\lambda-\mathcal{A}^{ij}_\lambda,\end{aligned}$$

where $A^{ij}_\lambda:=(1-\lambda)A^{ij}+\lambda\delta\cdot\delta_{ij}I,\ \mathcal{A}^{ij}_\lambda:=(1-\lambda)\mathcal{A}^{ij}$. Then we have

$$|A^{ij}_\lambda|\le|A^{ij}|,\quad |\mathcal{A}^{ij}_\lambda|\le|\mathcal{A}^{ij}|,\quad \delta|\xi|^2\le\sum_{i,j}\xi^*_iE^{ij}_\lambda\xi_j$$

for any real $d_1\times d$-matrix ξ. Also, we define

$$\bar{B}^i:=(1-\lambda)\bar{B}^i,\quad B^i_\lambda:=(1-\lambda)B^i,\quad C_\lambda:=(1-\lambda)C,\quad \mathcal{N}_\lambda:=(1-\lambda)\mathcal{N}.$$

Then $\bar{B}^i_\lambda,B^i_\lambda,C_\lambda,\mathcal{N}_\lambda$ satisfy (2.5). Thus, having the method of continuity in mind, we only prove that (2.6) and (2.7) hold given that a solution u already exists.

3. Applying the stochastic product rule $d|u^k|^2=2u^kdu^k+du^kdu^k$ for each k, we have

$$\begin{aligned}|u^k(t)|^2 &= |u^k_0|^2\\ &\quad+\int_0^t 2u^k\left(D_i(a^{ij}_{kr}u^r_{x^j}+\bar{f}^{ik})+b^i_{kr}u^r_{x^i}+c_{kr}u^r+f^k\right)ds\\ &\quad+\int_0^t|\sigma^i_{kr}u^r_{x^i}+\nu_{kr}u^r+g^k|^2_{\ell_2}ds\\ &\quad+\int_0^t 2u^k(\sigma^i_{kr,m}u^r_{x^i}+\nu_{kr,m}u^r+g^k_m)dw^m_s,\quad t>0.\end{aligned}\tag{2.8}$$

Note that, making the summation on r,i appeared, we get

$$\begin{aligned}&\sum_k\left|\sum_{r,i}\sigma^i_{kr}u^r_{x^i}+\sum_r\nu_{kr}u^r+g^k\right|^2_{\ell_2}\\ =\;& 2\sum_{i,j}(u_{x^i})^*\mathcal{A}^{ij}u_{x^j}+\sum_k\left[\left|(\mathcal{N}u)^k\right|^2_{\ell_2}+|g^k|^2_{\ell_2}\right]\\ &+2\sum_k\left[(\sum_i(\Sigma^iu_{x^i})^k,g^k)_{\ell_2}+((\mathcal{N}u)^k,g^k)_{\ell_2}+(\sum_i(\Sigma^iu_{x^i})^k,(\mathcal{N}u)^k)_{\ell_2}\right].\end{aligned}$$

By taking expectation, integrating with respect to x, and using integrating by parts in turn on (2.8), we obtain

$$\begin{aligned}
& \mathbb{E}\int_{\mathbb{R}^d} |u(t)|^2 dx + 2\,\mathbb{E}\int_0^t \int_{\mathbb{R}^d} \sum_{i,j} (u_{x^i})^* (A^{ij} - \mathcal{A}^{ij}) u_{x^j}\, dxds \\
= \; & \mathbb{E}\int_{\mathbb{R}^d} |u_0|^2 dx \\
& + 2\sum_i \mathbb{E}\int_0^t \int_{\mathbb{R}^d} \left[-2u^*_{x^i}\bar{f}^i + u^*(B^i u_{x^i})\right] dxds + 2\mathbb{E}\int_0^t \int_{\mathbb{R}^d} \left[Cu + u^* f\right] dxds \\
& + \sum_k \mathbb{E}\int_0^t \int_{\mathbb{R}^d} \left[\left|(\mathcal{N}u)^k\right|^2_{\ell_2} + |g^k|^2_{\ell_2}\right] dxds \\
& + 2\sum_k \mathbb{E}\int_0^t \int_{\mathbb{R}^d} \left[(\sum_i (\Sigma^i u_{x^i})^k, g^k)_{\ell_2} + ((\mathcal{N}u)^k, g^k)_{\ell_2} + (\sum_i (\Sigma^i u_{x^i})^k, (\mathcal{N}u)^k)_{\ell_2}\right] dxds.
\end{aligned}$$

Note that we have

$$\begin{aligned}
2\left|\sum_k (\sum_i (\Sigma^i u_{x^i})^k, g^k)_{\ell_2}\right| \;&\leq\; 2\sum_k \big|\sum_{r,i} \sigma^i_{kr} u^r_{x^i}\big|_{\ell_2} \left|g^k\right|_{\ell_2} \\
&\leq\; \sum_k \left(\frac{\varepsilon}{2}\big|\sum_{r,i} \sigma^i_{kr} u^r_{x^i}\big|^2_{\ell_2} + \frac{2}{\varepsilon}\left|g^k\right|^2_{\ell_2}\right) \\
&\leq\; \frac{\varepsilon}{2}|u_x|^2 \sum_{k,r,i} \left|\sigma^i_{kr}\right|^2_{\ell_2} + \frac{2}{\varepsilon}\sum_k \left|g^k\right|^2_{\ell_2} \\
&=\; \varepsilon|u_x|^2 \sum_{r,i} \left|\alpha^{ii}_{rr}\right|^2 + \frac{2}{\varepsilon}\sum_k \left|g^k\right|^2_{\ell_2} \qquad (2.9)
\end{aligned}$$

for any $\varepsilon > 0$; similarly, we get

$$2\left|\sum_k ((\mathcal{N}u)^k, g^k)_{\ell_2}\right| \;\leq\; |\mathcal{N}||u| + \sum_k |g^k|^2_{\ell_2}, \qquad (2.10)$$

$$2\left|\sum_k (\sum_i (\Sigma^i u_{x^i})^k, (\mathcal{N}u)^k)_{\ell_2}\right| \;\leq\; \varepsilon|u_x|^2 \sum_{r,i} \left|\alpha^{ii}_{rr}\right|^2 + \frac{2}{\varepsilon}|\mathcal{N}||u|. \qquad (2.11)$$

Hence, it follows that

$$\begin{aligned}
& \mathbb{E}\int_{\mathbb{R}^d} |u(t)|^2 dx + 2\delta\,\mathbb{E}\int_0^t \int_{\mathbb{R}^d} |u_x|^2 dxds \\
\leq\; & \mathbb{E}\int_{\mathbb{R}^d} |u_0|^2 dx + c\varepsilon\,\mathbb{E}\int_0^t \int_{\mathbb{R}^d} |u_x|^2 dxds + c\mathbb{E}\int_0^t \int_{\mathbb{R}^d} |u(s)|^2 dxds \\
& + c\sum_i \mathbb{E}\int_0^t \int_{\mathbb{R}^d} |\bar{f}^i|^2 dxds + \mathbb{E}\int_0^t \int_{\mathbb{R}^d} |f|^2 dxds + c\,\mathbb{E}\sum_k \int_0^t \int_{\mathbb{R}^d} |g^k|^2_{\ell_2} dxds \\
\leq\; & c\varepsilon\,\mathbb{E}\int_0^t \int_{\mathbb{R}^d} |u_x|^2 dxds + c\mathbb{E}\int_0^t \int_{\mathbb{R}^d} |u(s)|^2 dxds \\
& + c\sum_i \|\bar{f}^i\|^2_{\mathbb{L}_2(T)} + \|f\|^2_{\mathbb{L}_2(T)} + c\|g\|^2_{\mathbb{L}_2(T,\ell_2)} + \|u_0\|^2_{U^1_2}.
\end{aligned}$$

Choosing small ε, we obtain

$$\begin{aligned}
\|u_x\|^2_{\mathbb{L}_2(T)} &\leq c(\|u\|^2_{\mathbb{L}_2(T)} + \|f\|^2_{\mathbb{L}_2(T)} + \sum_i \|\bar{f}^i\|^2_{\mathbb{L}_2(T)} + \|g\|^2_{\mathbb{L}_2(T,\ell_2)} + \|u_0\|^2_{U^1_2}), \\
\mathbb{E}\int_{\mathbb{R}^d} |u(t)|^2 dx &\leq c\mathbb{E}\int_0^t \int_{\mathbb{R}^d} |u(s)|^2 dx ds \\
&\quad + c(\|f\|^2_{\mathbb{L}_2(T)} + \sum_i \|\bar{f}^i\|^2_{\mathbb{L}_2(T)} + \|g\|^2_{\mathbb{L}_2(T,\ell_2)} + \|u_0\|^2_{U^1_2}),
\end{aligned}$$

where c does not depend on T. Now we recall the remark in step 1, and see that the first inequality implies (2.6). Also the second inequality and Gronwall's inequality lead us to (2.7). The theorem is proved. □

3 The system on $\mathbb{R}^d_+$

In this section we present some results for the systems defined on $\mathbb{R}^d_+$. In the next section, these results will be modified and be used to develop our theory of the systems defined on C^1-domains.

Here we use the Banach spaces introduced in [11]. Let $\zeta \in C_0^\infty(\mathbb{R}_+)$ be a function satisfying

$$\sum_{n=-\infty}^{\infty} \zeta(e^{n+x}) > c > 0, \quad \forall x \in \mathbb{R}, \tag{3.1}$$

where c is a constant. Note that any nonnegative function ζ, $\zeta > 0$ on $[1,e]$, satisfies (3.1). For $\theta, \gamma \in \mathbb{R}$, we let $H^\gamma_{p,\theta}$ denote the set of all distributions $u = (u^1, u^2, \cdots u^{d_1})$ on $\mathbb{R}^d_+$ such that

$$\|u\|^p_{H^\gamma_{p,\theta}} := \sum_{n\in\mathbb{Z}} e^{n\theta} \|\zeta(\cdot)u(e^n\cdot)\|^p_{H^\gamma_p} < \infty. \tag{3.2}$$

If $g = (g^1, g^2, \ldots, g^{d_1})$ and each g^k is an ℓ_2-valued function, then we define

$$\|g\|^p_{H^\gamma_{p,\theta}(\ell_2)} = \sum_{n\in\mathbb{Z}} e^{n\theta} \|\zeta(\cdot)g(e^n\cdot)\|^p_{H^\gamma_p(\ell_2)}.$$

It is known (see [11]) that up to equivalent norms the space $H^\gamma_{p,\theta}$ is independent of the choice of ζ. Also, for any $\eta \in C_0^\infty(\mathbb{R}_+)$, we have

$$\sum_{n=-\infty}^{\infty} e^{n\theta} \|u(e^n\cdot)\eta\|^p_{H^\gamma_p} \leq c \sum_{n=-\infty}^{\infty} e^{n\theta} \|u(e^n\cdot)\zeta\|^p_{H^\gamma_p}, \tag{3.3}$$

where c depends only on $d, d_1, \gamma, \theta, p, \eta, \zeta$. Furthermore, if γ is a nonnegative integer, then

$$\|u\|^p_{H^\gamma_{p,\theta}} \sim \sum_{n=0}^{\gamma} \sum_{|\alpha|=n} \int_{\mathbb{R}^d_+} |(x^1)^n D^\alpha u(x)|^p (x^1)^{\theta-d}\, dx. \tag{3.4}$$

Below we collect some other properties of spaces $H^\gamma_{p,\theta}$. Let M^α be the operator of multiplying by $(x^1)^\alpha$ and $M = M^1$.

Lemma 3.1. ([11]) *Let $d-1<\theta<d-1+p$.*

(i) Assume that $\gamma-d/p=m+\nu$ for some $m=0,1,\cdots$ and $\nu\in(0,1]$. Then for any $u\in H^{\gamma}_{p,\theta}$ and $i\in\{0,1,\cdots,m\}$, we have

$$|M^{i+\theta/p}D^iu|_C+[M^{m+\nu+\theta/p}D^mu]_{C^\nu}\le c\|u\|_{H^{\gamma}_{p,\theta}}.$$

(ii) Let $\alpha\in\mathbb{R}$, then $M^{\alpha}H^{\gamma}_{p,\theta+\alpha p}=H^{\gamma}_{p,\theta}$,

$$\|u\|_{H^{\gamma}_{p,\theta}}\le c\|M^{-\alpha}u\|_{H^{\gamma}_{p,\theta+\alpha p}}\le c\|u\|_{H^{\gamma}_{p,\theta}}.$$

(iii) $MD,DM:H^{\gamma}_{p,\theta}\to H^{\gamma-1}_{p,\theta}$ are bounded linear operators.

(iv) There is a constant $c=c(d,p,\theta,\gamma)>0$ so that

$$c^{-1}\|M^{-1}u\|_{H^{\gamma}_{p,\theta}}\le\|u_x\|_{H^{\gamma-1}_{p,\theta}}\le c\|M^{-1}u\|_{H^{\gamma}_{p,\theta}}.$$

We define the following stochastic Banach spaces.

$$\mathbb{H}^{\gamma}_{p,\theta}(T)=L_p(\Omega\times[0,T],\mathcal{P},H^{\gamma}_{p,\theta}),\quad \mathbb{H}^{\gamma}_{p,\theta}(T,\ell_2)=L_p(\Omega\times[0,T],\mathcal{P},H^{\gamma}_{p,\theta}(\ell_2))$$

$$\mathbb{L}_{p,\theta}(T):=\mathbb{H}^{0}_{p,\theta}(T),\quad \mathbb{L}_{p,\theta}(T,\ell_2):=\mathbb{H}^{0}_{p,\theta}(T,\ell_2),\quad U^{\gamma}_{p,\theta}=L_p(\Omega,\mathcal{F}_0,M^{1-2/p}H^{\gamma-2/p}_{p,\theta}).$$

Definition 3.2. We write $u\in\mathfrak{H}^{\gamma+2}_{p,\theta}(T)$ if $u\in M\mathbb{H}^{\gamma+2}_{p,\theta}(T)$, $u(0)\in U^{\gamma+2}_{p,\theta}$ and for some $f\in M^{-1}\mathbb{H}^{\gamma}_{p,\theta}(T)$, $g\in\mathbb{H}^{\gamma+1}_{p,\theta}(T,\ell_2)$,

$$du=fdt+g_mdw^m_t$$

holds in the sense of the distributions. The norm in $\mathfrak{H}^{\gamma+2}_{p,\theta}(T)$ is defined by

$$\|u\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(T)}=\|M^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(T)}+\|Mf\|_{\mathbb{H}^{\gamma}_{p,\theta}(T)}+\|g\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(T,\ell_2)}+\|u(0)\|_{U^{\gamma+2}_{p,\theta}}.$$

Let us denote

$$K:=\sqrt{\sum_j(K^j)^2}.$$

Lemma 3.3. *Assume*

$$\theta\in\left(d-\frac{\delta}{2K-\delta},\ d+\frac{\delta}{2K+\delta}\right),\tag{3.5}$$

$\bar{b}^i=b^i=c=0$ and $\nu=0$. Then if $u\in M\mathbb{H}^1_{2,\theta}(T)$ is a solution of system (1.1) on $[0,T]\times\mathbb{R}^d_+$ and $u\in L_2(\Omega,C([0,T],C^1_0((1/N,N)\times\{x':|x'|<N\})))$ for some $N>0$, then we have

$$\|M^{-1}u\|^2_{\mathbb{H}^1_{2,\theta}(T)}\le c(\|\bar{f}^i\|_{\mathbb{L}_{2,\theta}(T)}+\|Mf\|^2_{\mathbb{H}^{-1}_{2,\theta}(T)}+\|g\|^2_{\mathbb{L}_{2,\theta}(T,\ell_2)}+\|u_0\|^2_{U^1_{2,\theta}}),\tag{3.6}$$

where $c=c(d,d_1,\delta,\theta,K,L)$.

Proof. 1. By Corollary 2.12 in [11], f^k has the following representation:

$$f^k=\sum_{i=1}^dD_iF^{ik},\quad \sum_i\|F^{ik}\|_{L_{2,\theta}}\le c\|Mf^k\|_{H^{-1}_{2,\theta}}.$$

Also since $\|M^{-1}u\|_{H^1_{2,\theta}} \le c\|u_x\|_{L_{2,\theta}}$ (see Lemma 3.1(iv)), it is enough to assume $f^k = 0$ and prove

$$\|u_x\|^2_{\mathbb{L}_{2,\theta}(T)} \le c(\|\bar{f}^i\|_{\mathbb{L}_{2,\theta}(T)} + \|g\|^2_{\mathbb{L}_{2,\theta}(T,\ell_2)} + \|u_0\|^2_{U^1_{2,\theta}}).$$

2. Again, as in the proof of Theorem 2.4, applying the stochastic product rule $d|u^k|^2 = 2u^k du^k + du^k du^k$ for each k, we get

$$\begin{aligned} |u^k(t)|^2 &= |u^k_0|^2 + \int_0^t 2u^k \left[D_i(a^{ij}_{kr}u^r_{x^j} + \bar{f}^{ik})\right] ds \\ &\quad + \int_0^t |\sigma^i_{kr}u^r_{x^i} + g^k|^2_{\ell_2} ds + \int_0^t 2u^k(\sigma^i_{kr,m}u^r_{x^i} + g^k_m) dw^m_s, \end{aligned}$$

where the summations on i, j, r are understood. Denote $c = \theta - d$. For each k, we have

$$\begin{aligned} 0 &\le \mathbb{E}\int_{\mathbb{R}^d_+} |u^k(T,x)|^2 (x^1)^c dx \\ &= \mathbb{E}\int_{\mathbb{R}^d_+} |u^k(0,x)|^2 (x^1)^c dx \\ &\quad + 2\mathbb{E}\int_0^T \int_{\mathbb{R}^d_+} u^k D_i(a^{ij}_{kr}u^r_{x^j})(x^1)^c dxds + 2\mathbb{E}\int_0^T \int_{\mathbb{R}^d_+} u^r \bar{f}^{ik}_{x^i}(x^1)^c dxds \\ &\quad + \mathbb{E}\int_0^T \int_{\mathbb{R}^d_+} |\sigma^i_{kr}u^r_{x^i}|^2_{\ell_2}(x^1)^c dxds \\ &\quad + 2\mathbb{E}\int_0^T \int_{\mathbb{R}^d_+} ((\Sigma^i u_{x^i})^k, g^k)_{\ell_2}(x^1)^c dxds + \mathbb{E}\int_0^T \int_{\mathbb{R}^d_+} |g^k|^2_{\ell_2}(x^1)^c dxds. \end{aligned} \tag{3.7}$$

Note that, by integration by parts, we get

$$\begin{aligned} 2\mathbb{E}\int_0^T \int_{\mathbb{R}^d_+} u^r \bar{f}^{ik}_{x^i}(x^1)^c dxds &= -2\mathbb{E}\int_0^T \int_{\mathbb{R}^d_+} \left[u^r_{x^i}\bar{f}^{ik}(x^1)^c + cM^{-1}u^r\bar{f}^{1k}(x^1)^c\right] dxds \\ &\le \varepsilon\|u_x\|^2_{\mathbb{L}_{2,\theta}(T)} + \varepsilon\|M^{-1}u\|^2_{\mathbb{L}_{2,\theta}(T)} + c(\varepsilon)\|\bar{f}\|^2_{\mathbb{L}_{2,\theta}(T)}. \end{aligned}$$

Also, the second term in the right hand side of (3.7) is

$$\mathbb{E}\int_0^T \int_{\mathbb{R}^d_+} \left[-2a^{ij}_{kr}u^k_{x^i}u^r_{x^j} - 2c(a^{1j}_{kr}u^r_{x^j})(M^{-1}u^k)\right](x^1)^c dxds.$$

Thus, by summing up the terms in (3.7) over k and rearranging the terms, we obtain

$$\begin{aligned} &2\mathbb{E}\int_0^T \int_{\mathbb{R}^d_+} u^*_{x^i}\left(A^{ij} - \mathcal{A}^{ij}\right) u_{x^j}\,(x^1)^c dxds \\ \le\ & |c|\left(\kappa\|u_x\|^2_{\mathbb{L}_{2,\theta}(T)} + K^2\kappa^{-1}\|M^{-1}u\|^2_{\mathbb{L}_{2,\theta}(T)}\right) \\ & + N\varepsilon\left(\|M^{-1}u\|^2_{\mathbb{L}_{2,\theta}(T)} + \|u_x\|^2_{\mathbb{L}_{2,\theta}(T)}\right) \\ & + c(\varepsilon)\left(\|\bar{f}^i\|^2_{\mathbb{L}_{2,\theta}(T)} + \|g\|^2_{\mathbb{L}_{2,\theta}(T,\ell_2)}\right) + \|u(0)\|^2_{U^1_{2,\theta}}, \end{aligned} \tag{3.8}$$

for any $\kappa, \varepsilon > 0$. This is because for any vectors $v, w \in \mathbb{R}^n$ and $\kappa > 0$

$$| < A^{1j}v, w > | \le |A^{1j}v||w| \le K^j|v||w| \le \frac{1}{2}(\kappa|v|^2 + \kappa^{-1}(K^j)^2|w|^2)$$

and consequently,

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_{\mathbb{R}^d_+}\left[-2a^{ij}_{kr}u^k_{x^i}u^r_{x^j}-2c(a^{1j}_{kr}u^r_{x^j})(M^{-1}u^k)\right](x^1)^c dxds\\
\le\ &\mathbb{E}\int_0^T\int_{\mathbb{R}^d_+}-2a^{ij}_{kr}u^k_{x^i}u^r_{x^j}\,dxds+|c|\left(\kappa\|u_x\|^2_{\mathbb{L}_{2,\theta}(T)}+K^2\kappa^{-1}\|M^{-1}u\|^2_{\mathbb{L}_{2,\theta}(T)}\right). \qquad (3.9)
\end{aligned}$$

Now, Assumption (2.3), inequality (3.8), the inequality

$$\|M^{-1}u\|^2_{L_{2,\theta}}\le\frac{4}{(d+1-\theta)^2}\|u_x\|^2_{L_{2,\theta}} \qquad (3.10)$$

(see Corollary 6.2 in [11]), and Lemma 3.1 (iv) lead us to

$$\begin{aligned}
&2\delta\|u_x\|^2_{\mathbb{L}_{2,\theta}(T)}-|c|\left(\kappa+\frac{4K^2}{\kappa(d+1-\theta)^2}\right)\|u_x\|^2_{\mathbb{L}_{2,\theta}(T)}\\
&\le N\varepsilon\|u_x\|^2_{\mathbb{L}_{2,\theta}(T)}+N\|\bar f^i\|^2_{\mathbb{L}_{2,\theta}(T)}+N\|g\|^2_{\mathbb{L}_{2,\theta}(T)}+\|u(0)\|^2_{U^1_{2,\theta}}.
\end{aligned}$$

Now, it is enough to take $\kappa=2K/(d+1-\theta)$ and observe that (3.5) is equivalent to the condition

$$2\delta-|c|\left(\kappa+\frac{4K}{\kappa(d+1-\theta)^2}\right)=2\delta-\frac{4|c|K}{d+1-\theta}>0.$$

The lemma is proved. □

Here is the main result of this section.

Theorem 3.4. *Suppose (3.5) holds and*

$$|M\bar b^i|+|Mb^i|+|M^2c|+|M\nu|_{\ell_2}<\beta. \qquad (3.11)$$

Then there exists constant $\beta_0=\beta_0(d,d_1,\theta,\delta,K,L)>0$ so that if $\beta\le\beta_0$, then for any $\bar f^i\in\mathbb{L}_{2,\theta}(T)$, $f\in M^{-1}\mathbb{H}^{-1}_{2,\theta}(T)$, $g\in\mathbb{L}_{2,\theta}(T,\ell_2)$, and $u_0\in U^1_{2,\theta}$, system (1.1) has a unique solution $u\in\mathfrak{H}^1_{2,\theta}(T)$, and furthermore

$$\|u\|_{\mathfrak{H}^1_{2,\theta}(T)}\le c\|\bar f^i\|_{\mathbb{L}_{2,\theta}(T)}+c\|Mf\|_{\mathbb{H}^{-1}_{2,\theta}(T)}+c\|g\|_{\mathbb{L}_{2,\theta}(T,\ell_2)}+c\|u_0\|_{U^1_{2,\theta}} \qquad (3.12)$$

where $c=c(d,\delta,K,L,T)$.

Proof. As before, we only prove that the a priori estimate (3.12) holds given that a solution u already exists. By Theorem 2.9 in [12], for any nonnegative integer $n\ge\gamma$, the set

$$\mathfrak{H}^n_{2,\theta}(T)\cap\bigcup_{N=1}^\infty L_2(\Omega,C([0,T],C^n_0((1/N,N)\times\{x':|x'|<N\})))$$

is everywhere dense in $\mathfrak{H}^\gamma_{2,\theta}(T)$ and thus we may assume that u is sufficiently smooth in x and vanishes near the boundary.

Step 1. If $\bar b^i=b^i=c=0$ and $\nu=0$ the a priori estimate follows from Lemma 3.3.

Step 2. In general, by Step 1,

$$\begin{aligned}\|M^{-1}u\|_{\mathbb{H}^1_{2,\theta}(T)} \quad &\leq \quad c\|M\bar{b}^iM^{-1}u+\bar{f}^i\|_{\mathbb{L}_{2,\theta}(T)}+c\|Mb^iu_{x^i}+M^2cM^{-1}u+Mf\|_{\mathbb{H}^{-1}_{2,\theta}(T)}\\ &\qquad +c\|M\nu M^{-1}u+g\|_{\mathbb{L}_{2,\theta}(T,\ell_2)}+c\|u_0\|_{U^1_{2,\theta}}.\end{aligned}$$

Since $\|\cdot\|_{H^{-1}_{2,\theta}} \leq \|\cdot\|_{L_{2,\theta}}$, we easily see that the above is less than

$$c\beta\|M^{-1}u\|_{\mathbb{H}^1_{2,\theta}(T)}+c\|\bar{f}\|_{\mathbb{L}_{2,\theta}(T)}+c\|Mf\|_{\mathbb{H}^{-1}_{2,\theta}(T)}+c\|g\|_{\mathbb{L}_{2,\theta}(T,\ell_2)}+c\|u_0\|_{U^1_{2,\theta}}.$$

Now it is enough to take β_0 so that $c\beta<1/2$ for any $\beta \leq \beta_0$. The theorem is proved.

□

Remark 3.5. We do not know how sharp (3.5) is. However, if $\theta \notin (d-1,d+1)$ then Theorem 3.4 is false even for the heat equation $u_t=\Delta u+f$ (see [11]).

We also mention that if the coefficients are sufficiently smooth in x, then one can get quite wider range of θ. This will be shown in the subsequent article [6].

4 The system on $\mathcal{O} \subset \mathbb{R}^d$

In this section we assume the following.

Assumption 4.1. The domain $\mathcal{O}$ is of class C^1_u. In other words, for any $x_0 \in \partial\mathcal{O}$, there exist constants $r_0, K_0 \in (0,\infty)$ and a one-to-one continuously differentiable mapping Ψ of $B_{r_0}(x_0)$ onto a domain $J \subset \mathbb{R}^d$ such that

(i) $J_+ := \Psi(B_{r_0}(x_0)\cap\mathcal{O}) \subset \mathbb{R}^d_+$ and $\Psi(x_0)=0$;

(ii) $\Psi(B_{r_0}(x_0)\cap\partial\mathcal{O}) = J\cap\{y\in\mathbb{R}^d : y^1=0\}$;

(iii) $\|\Psi\|_{C^1(B_{r_0}(x_0))} \leq K_0$ and $|\Psi^{-1}(y_1)-\Psi^{-1}(y_2)| \leq K_0|y_1-y_2|$ for any $y_i \in J$;

(iv) Ψ_x is uniformly continuous in for $B_{r_0}(x_0)$.

To proceed further we introduce some well known results from [2] and [8].

Lemma 4.2. *Let the domain $\mathcal{O}$ be of class C^1_u. Then*

(i) there is a bounded real-valued function ψ defined in $\bar{\mathcal{O}}$ such that the functions $\psi(x)$ and $\rho(x) := dist(x,\partial\mathcal{O})$ are comparable in the part of a neighborhood of $\partial\mathcal{O}$ lying in $\mathcal{O}$. In other words, if $\rho(x)$ is sufficiently small, say $\rho(x) \leq 1$, then $N^{-1}\rho(x) \leq \psi(x) \leq N\rho(x)$ with some constant N independent of x,

(ii) for any multi-index α it holds that

$$\sup_{\mathcal{O}} \psi^{|\alpha|}(x)|D^\alpha\psi_x(x)| < \infty. \tag{4.1}$$

To describe the assumptions of $\bar{f}^i$s, f, and g in (1.1) with space domain $\mathcal{O}$ we use the Banach spaces introduced in [8] and [13]. Let $\zeta \in C^\infty_0(\mathbb{R}_+)$ be a nonnegative function satisfying (3.1). For $x\in\mathcal{O}$ and $n\in\mathbb{Z} := \{0,\pm1,...\}$ we define

$$\zeta_n(x)=\zeta(e^n\psi(x)).$$

Then we have $\sum_n \zeta_n \geq c$ in $\mathcal{O}$ and

$$\zeta_n \in C_0^\infty(\mathcal{O}), \quad |D^m \zeta_n(x)| \leq N(m) e^{mn}.$$

For $\theta, \gamma \in \mathbb{R}$, let $H^\gamma_{p,\theta}(\mathcal{O})$ denote the set of all distributions $u = (u^1, u^2, \cdots u^{d_1})$ on $\mathcal{O}$ such that

$$\|u\|^p_{H^\gamma_{p,\theta}(\mathcal{O})} := \sum_{n\in\mathbb{Z}} e^{n\theta} \|\zeta_{-n}(e^n \cdot) u(e^n \cdot)\|^p_{H^\gamma_p} < \infty. \tag{4.2}$$

If $g = (g^1, g^2, \dots, g^{d_1})$ and each g^k is an ℓ_2-valued function, then we define

$$\|g\|^p_{H^\gamma_{p,\theta}(\mathcal{O},\ell_2)} = \sum_{n\in\mathbb{Z}} e^{n\theta} \|\zeta_{-n}(e^n \cdot) g(e^n \cdot)\|^p_{H^\gamma_p(\ell_2)}.$$

It is known (see, for instance, [13]) that up to equivalent norms the space $H^\gamma_{p,\theta}(\mathcal{O})$ is independent of the choice of ζ and ψ. Moreover, if $\gamma = n$ is a non-negative integer, then it holds that

$$\|u\|^p_{H^\gamma_{p,\theta}(\mathcal{O})} \sim \sum_{k=0}^{n} \sum_{|\alpha|=k} \int_{\mathcal{O}} |\psi^k D^\alpha u(x)|^p \psi^{\theta-d}(x)\, dx. \tag{4.3}$$

By comparing (3.4) and (4.3), one finds that two spaces $H^\gamma_{p,\theta}(\mathbb{R}^d_+)$ and $H^\gamma_{p,\theta}$ are different since ψ is bounded. Also, it is easy to see that, for any nonnegative function $\xi = \xi(x^1) \in C_0^\infty(\mathbb{R}^1)$ satisfying $\xi = 1$ near $x^1 = 0$, we have

$$\|u\|_{H^\gamma_{p,\theta}(\mathbb{R}^d_+)} \sim \left(\|\xi u\|_{H^\gamma_{p,\theta}} + \|(1-\xi)u\|_{H^\gamma_p} \right). \tag{4.4}$$

In particular, if $u(x) = 0$ for $x \geq r$, then for any $\alpha \in \mathbb{R}$ we get

$$c^{-1} \|M^\alpha u\|_{H^\gamma_{p,\theta}} \leq \|\psi^\alpha u\|_{H^\gamma_{p,\theta}(\mathbb{R}^d_+)} \leq c \|M^\alpha u\|_{H^\gamma_{p,\theta}}, \tag{4.5}$$

where $c = c(r, \alpha, \gamma, p, \theta)$. We also mention that the space $H^\gamma_{p,\theta}$ can be defined on the basis of (4.2) by formally taking $\psi(x) = x^1$ so that $\zeta_{-n}(e^n x) = \zeta(x)$ and (4.2) becomes

$$\|u\|^p_{H^\gamma_{p,\theta}} := \sum_{n\in\mathbb{Z}} e^{n\theta} \|u(e^n \cdot)\zeta\|^p_{H^\gamma_p} < \infty.$$

We place the following lemma similar to Lemma 3.1.

Lemma 4.3. ([11]) *Let $d - 1 < \theta < d - 1 + p$.*

Assertions (i)-(iii) in Lemma 3.1 hold true with ψ and $H^\gamma_{p,\theta}(\mathcal{O})$ in place of M and $H^\gamma_{p,\theta}$, respectively.

We define

$$\mathbb{H}^\gamma_{p,\theta}(\mathcal{O}, T) = L_p(\Omega \times [0,T], \mathcal{P}, H^\gamma_{p,\theta}(\mathcal{O})), \quad \mathbb{H}^\gamma_{p,\theta}(\mathcal{O}, T, \ell_2) = L_p(\Omega \times [0,T], \mathcal{P}, H^\gamma_{p,\theta}(\mathcal{O}, \ell_2)),$$

$$U^\gamma_{p,\theta}(\mathcal{O}) = \psi^{1-2/p} L_p(\Omega, \mathcal{F}_0, H^{\gamma-2/p}_{p,\theta}(\mathcal{O})), \quad \mathbb{L}_{p,\theta}(\mathcal{O}, T) = \mathbb{H}^0_{p,\theta}(\mathcal{O}, T).$$

Definition 4.4. We define $\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)$ as the space of all functions $u=(u^1,\cdots,u^{d_1})\in\psi\mathbb{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)$ such that $u(0,\cdot)\in U^{\gamma+2}_{p,\theta}(\mathcal{O})$ and for some $f\in\psi^{-1}\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)$, $g\in\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)$,

$$du=f\,dt+g_m\,dw^m_t,$$

in the sense of distributions. The norm in $\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)$ is introduced by

$$\|u\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)}=\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)}+\|\psi f\|_{\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)}+\|g\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T,\ell_2)}+\|u(0,\cdot)\|_{U^{\gamma+2}_{p,\theta}(\mathcal{O})}.$$

The following result is due to N.V.Krylov (see, for instance, [9]).

Lemma 4.5. *Let $p\geq 2$. Then there exists a constant $c=c(d,p,\theta,\gamma,T)$ such that*

$$\mathbb{E}\sup_{t\leq T}\|u(t)\|^p_{H^{\gamma+1}_{p,\theta}(\mathcal{O})}\leq c\|u\|^p_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)}.$$

In particular, for any $t\leq T$,

$$\|u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},t)}\leq c\int_0^t\|u\|^p_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},s)}ds.$$

Assumption 4.6. There is control on the behavior of $\bar{b}^i_{kr}$, b^i_{kr}, c_{kr} and ν_{kr} near $\partial\mathcal{O}$, namely,

$$\lim_{\substack{\rho(x)\to 0\\ x\in\mathcal{O}}}\sup_{t,\omega}[\rho(x)|\bar{b}^i_{kr}(t,x)|+\rho(x)|b^i_{kr}(t,x)|+\rho^2(x)|c_{kr}(t,x)|+\rho(x)|\nu_{kr}(t,x)|_{\ell_2}]=0. \tag{4.6}$$

Note that Assumption 4.6 allows the coefficients to be unbounded and to blow up near the boundary. (4.6) holds if, for instance,

$$|\bar{b}^i_{kr}(t,x)|+|b^i_{kr}(x)|+|\nu_{kr}(x)|_{\ell_2}\leq c\rho^{-1+\varepsilon}(x),\quad |c_{kr}(t,x)|\leq\rho^{-2+\varepsilon}(x),$$

for some $c,\varepsilon>0$.

Here is the main result of this section.

Theorem 4.7. *Let $\mathcal{O}=\mathbb{R}^d_+$ or $\mathcal{O}$ be bounded. Suppose (3.5) and Assumption 4.6 hold. Then for any $\bar{f}^i\in\mathbb{L}_{2,\theta}(\mathcal{O},T)$ (i=1,...,d), $f\in\psi^{-1}\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},T)$, $g\in\mathbb{L}_{2,\theta}(\mathcal{O},T,\ell_2)$, and $u_0\in U^1_{2,\theta}(\mathcal{O})$, the system (1.1) admits a unique solution $u\in\mathfrak{H}^1_{2,\theta}(\mathcal{O},T)$, and for this solution we have*

$$\|\psi^{-1}u\|_{\mathbb{H}^1_{2,\theta}(\mathcal{O},T)}\leq c\|\bar{f}^i\|_{\mathbb{L}_{2,\theta}(\mathcal{O},T)}+c\|\psi f\|_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},T)}+c\|g\|_{\mathbb{L}_{2,\theta}(\mathcal{O},T,\ell_2)}+c\|u_0\|_{U^1_{2,\theta}(\mathcal{O})}, \tag{4.7}$$

where $c=c(d,\delta,\theta,K,L)$.

Remark 4.8. By carefully inspecting our arguments below one can check that Theorem 4.7 holds even if the C^1-domain $\mathcal{O}$ is not bounded.

To prove Theorem 4.7 we need the following a priori estimate near the boundary.

Lemma 4.9. *Suppose that $u\in\mathfrak{H}^1_{2,\theta}(\mathcal{O},T)$ is a solution of system (1.1) such that $u(t,x)=0$ for $x\in\mathcal{O}\backslash B_r(x_0)$, $x_0\in\partial\mathcal{O}$. Then there exists constant $r_1\in(0,1)$, independent of x_0 and u, such that if $r\leq r_1$, then a priori estimate (4.7) holds.*

Proof. Let $x_0 \in \partial\mathcal{O}$ and Ψ be a function from Assumption 4.1. In [8] it is shown that Ψ can be chosen in such a way that

$$\rho(x)\Psi_{xx}(x) \to 0 \quad \text{as} \quad x \in B_{r_0}(x_0) \cap \mathcal{O}, \text{and } \rho(x) \to 0, \tag{4.8}$$

where the convergence in (4.8) is independent of x_0.

Define $r = r_0/K_0$ and fix smooth functions $\eta \in C_0^\infty(B_r), \varphi \in C^\infty(\mathbb{R})$ such that $0 \le \eta, \varphi \le 1$, and $\eta = 1$ in $B_{r/2}$, $\varphi(t) = 1$ for $t \le -3$, and $\varphi(t) = 0$ for $t \ge -1$ and $0 \ge \varphi' \ge -1$. We observe that $\Psi(B_{r_0}(x_0))$ contains B_r. For $n = 1, 2, ...,\ t > 0,\ x \in \mathbb{R}^d_+$ we introduce $\varphi_n(x) := \varphi(n^{-1} \ln x^1)$,

$$\begin{aligned}
\hat{a}^{ij}(t,x) \quad &:= \quad \eta(x)\left(\sum_{l,m=1}^{d} a^{lm}(t, \Psi^{-1}(x)) \cdot \partial_l \Psi^i(\Psi^{-1}(x)) \cdot \partial_m \Psi^j(\Psi^{-1}(x))\right) + \delta^{ij}(1-\eta(x))I,\\
\hat{\bar{b}}^{i,n}(t,x) \quad &:= \quad \eta(x)\varphi_n(x)\sum_l \bar{b}^l(t, \Psi^{-1}(x)) \cdot \partial_l \Psi^i(\Psi^{-1}(x)),\\
\hat{b}^{i,n}(t,x) \quad &:= \quad \eta(x)\varphi_n(x)\Big[-\sum_{l,m,r,j} a^{lm}(t, \Psi^{-1}(x)) \cdot (\partial_m \Psi^j \cdot \partial_{lr}\Psi^i)(\Psi^{-1}(x)) \cdot \partial_j(\Psi^{-1})^r(x)\\
&\qquad\qquad + \sum_l b^l(t, \Psi^{-1}(x)) \cdot \partial_l \Psi^i(\Psi^{-1}(x))\Big],\\
\hat{c}^n(t,x) \quad &:= \quad \eta(x)\varphi_n(x)c(t, \Psi^{-1}(x)),\\
\hat{\sigma}^i(t,x) \quad &:= \quad \eta(x)\sum_l \sigma^l(t, \Psi^{-1}(x)) \cdot \partial_l \Psi^i(\Psi^{-1}(x)),\\
\hat{\nu}^n(t,x) \quad &:= \quad \eta(x)\varphi_n(x)\nu(t,x)(t, \Psi^{-1}(x)).
\end{aligned}$$

Then $(\hat{a}^{ij}, \hat{\sigma}^i)$ satisfies (2.3) and (2.4). We take β_0 from Theorem 3.4 corresponding to $d, d_1, \theta, \delta, L$ and K. We observe that $\varphi_n(x) = 0$ for $x^1 \ge e^{-n}$. Also, note that (4.8) implies $x^1\Psi_{xx}(\Psi^{-1}(x)) \to 0$ as $x^1 \to 0$. Using these facts and (4.6), one can fix $n > 0$ which is sufficiently large, independent of x_0, and

$$x^1|\hat{\bar{b}}^{i,n}_{kr}(t,x)| + x^1|\hat{b}^{i,n}_{kr}(t,x)| + (x^1)^2|\hat{c}^n_{kr}(t,x)| + x^1|\hat{\nu}^n_{kr}(t,x)|_{\ell_2} \le \beta_0, \quad \forall\ \omega, t, x.$$

Now, we fix $r_1 < r_0$ so that

$$\Psi(B_{r_1}(x_0)) \subset B_{r/2} \cap \{x : x^1 \le e^{-3n}\}. \tag{4.9}$$

Next, we observe that, by Lemma 4.2 and Theorem 3.2 in [13] (or see [8]), for any $\nu, \alpha \in \mathbb{R}$ and $h \in \psi^{-\alpha} H^\nu_{p,\theta}(\mathcal{O})$ with support in $B_{r_0}(x_0)$ we have

$$\|\psi^\alpha h\|_{H^\nu_{p,\theta}(\mathcal{O})} \sim \|M^\alpha h(\Psi^{-1})\|_{H^\nu_{p,\theta}}. \tag{4.10}$$

Thus, for $v(t,x) := u(t, \Psi^{-1}(x))$ we have $u \in \mathfrak{H}^1_{2,\theta}(T)$ and v satisfies

$$\begin{aligned}
dv^k \quad &= \quad (D_i(\hat{a}^{ij}_{kr} v^r_{x^j} + \hat{\bar{b}}^{i,n}_{kr} v^r + \hat{\bar{f}}^{ik}) + \hat{b}^{i,n}_{kr} v^r_{x^i} + \hat{c}^n_{kr} v^r + \hat{f}^k)dt\\
&\qquad + (\hat{\sigma}^i_{kr,m} v^r_{x^i} + \hat{\nu}^n_{kr,m} v^r + \hat{g}^k_m)dw^m_t,
\end{aligned}$$

where

$$\hat{\bar{f}}^{ik} = \sum_{\ell} (\bar{f}^{ik} \partial_i \Psi^\ell)(\Psi^{-1}(x)), \quad \hat{f}^k = \bar{f}^{ik}(\Psi^{-1}(x))\partial_{ij}\Psi^\ell(\Psi^{-1}(x))\partial_i(\Psi^{-1})^j(x) + f^k(\Psi^{-1}(x)).$$

Hence, the a priori estimate follows from Theorem 3.4 and (4.10). The lemma is proved.

□

Remark 4.10. Let $\mathcal{O} = \mathbb{R}^d_+$. Then, in fact, Lemma 4.9 holds if $u(t,x) = 0$ for $x^1 \geq r_1$ for some r_1. Indeed, by (4.6) there is $r_1 > 0$ so that

$$|M\bar{b}^i| + |Mb^i| + |M^2 c| + |M\nu|_{\ell_2} < \beta_0 \tag{4.11}$$

for $x^1 \leq r_1$. Now, if $u(t,x) = 0$ for $x^1 \geq r_1$, then without affecting the system we may put $\bar{b}^i = b^i = c = 0$ and $\nu = 0$ for $x^1 \geq r_1$ so that (4.11) holds for all x. Consequently the assertion follows from Theorem 3.4 and (4.5).

Next, we prove the a priori estimate for small T.

Lemma 4.11. *Let assumptions in Theorem 4.7 be satisfied. Then there exists a constant $\varepsilon \in (0,1)$ so that if $T \leq \varepsilon$, then a priori estimate (4.7) holds for any solution $u \in \mathfrak{H}^1_{2,\theta}(\mathcal{O},T)$ of system (1.1) with $u_0 = 0$.*

Proof. We prove the lemma only when $\mathcal{O}$ is bounded. The case $\mathcal{O} = \mathbb{R}^d_+$ is treated similarly. Take a partition of unity $\{\zeta_n : n = 0,1,2,...,N_0\}$, where $N_0 < \infty$, such that $\zeta_0 \in C^\infty_0(\mathcal{O})$ and $\zeta_n \in C^\infty_0(B_{r_1/2}(x_n))$ with $x_n \in \partial\mathcal{O}$ for $n = 1,\dots,N_0$. Also, we fix functions $\bar{\zeta}_n$ such that $\bar{\zeta}_0 \in C^\infty_0(\mathcal{O}), \bar{\zeta}_n \in C^\infty_0(B_{r_1}(x_n))$ for $n = 1,\dots,N_0$, and $\zeta_n\bar{\zeta}_n = \zeta_n$ for each n. We note that $v_n := u\zeta_n$ satisfies

$$\begin{aligned} dv^k_n \quad &= \quad (D_i(a^{ij}_{kr}v^r_{nx^j} + \bar{b}^i_{kr}v^r_n + \bar{f}^{ik}_n) + b^i_{kr}v^r_{nx^i} + c_{kr}v^r_n + f^k_n - a^{ij}_{kr}u^r_{x^j}\zeta_{nx^i})\,dt \\ &\quad +(\sigma^{ik}_{kr,m}v^r_{nx^i} + \nu_{kr,m}v^r_n + g^k_m)\,dw^m_t, \end{aligned} \tag{4.12}$$

where

$$f^k_n := -(\bar{b}^i_{kr}u^r + \bar{f}^{ik} + b^i_{kr}u^r)\zeta_{nx^i} + f^k\zeta_n,$$

$$\bar{f}^{ik}_n := -a^{ij}_{kr}u^r\zeta_{nx^j} + \bar{f}^{ik}\zeta_n, \qquad g^k_n = -\sigma^{ik}u\zeta_{nx^i} + g^k\zeta_n.$$

Also, we note that $\zeta_0 u \in \mathcal{H}^1_2(T)$ and $\|\psi^{-1}\zeta_0 u\|_{\mathbb{H}^1_{2,\theta}(\mathcal{O},T)} \sim \|\zeta_0 u\|_{\mathbb{H}^1_2(T)}$. By Theorem 2.4 and Lemma 4.9, we have

$$\|\psi^{-1}u\|^2_{\mathbb{H}^1_{2,\theta}(\mathcal{O},T)} \leq \sum_{n=0}^{N_0} \|\psi^{-1}v_n\|^2_{\mathbb{H}^1_{2,\theta}(\mathcal{O},T)} \tag{4.13}$$

$$\leq N\sum_{n=0}^{N_0}(\|\bar{f}^i_n\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},T)} + \|\psi f_n\|^2_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},T)} + \|\psi a^{ij}_{kr}u^r_x\zeta_{nx}\|^2_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},T)} + \|g_n\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},T,\ell_2)}). \tag{4.14}$$

Actually relations like (4.13) hold even if $N_0 = \infty$ and this is why the theorem is true even when $\mathcal{O}$ is not bounded.

Since a^{ij} is only measurable, at most we get

$$\sum_n \|\psi a^{ij}_{kr} u^r_{x^j}\zeta_{nx^i}\|^2_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},T)} \leq \sum_n \|\psi a^{ij}_{kr} u^r_{x^j}\zeta_{nx^i}\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},T)} \leq N\|u_x\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},T)} \leq N\|\psi^{-1}u\|^2_{\mathbb{H}^1_{2,\theta}(\mathcal{O},T)}$$

and consequently (4.14) only leads us to the useless inequality

$$\|\psi^{-1}u\|^2_{\mathbb{H}^1_{2,\theta}(\mathcal{O},T)} \leq N\|\psi^{-1}u\|^2_{\mathbb{H}^1_{2,\theta}(\mathcal{O},T)} + \cdots.$$

Hence, to avoid estimating the norm $\|\psi a^{ij}_{kr} u^r_{x^j}\zeta_{nx^i}\|_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},T)}$ we proceed as in [5]. We note that for each k we have

$$\psi^{-1} a^{ij}_{kr} u^r_{x^j}\zeta_{nx^i} \in \psi^{-1}\mathbb{L}_{2,\theta}(\mathcal{O},T).$$

Thus, by Theorem 2.9 in [4], for each k the solution $\bar{v}^k_n \in \mathfrak{H}^2_{2,\theta}(\mathcal{O},T)$ of the single equation

$$dv = (\Delta v - \psi^{-1} a^{ij}_{kr} u^r_{x^j}\zeta_{nx^i})dt, \quad v(0) = 0$$

satisfies

$$\|\bar{v}^k_n\|_{\mathfrak{H}^2_{2,\theta}(\mathcal{O},T)} \leq N\|a^{ij}_{kr} u^r_{x^j}\zeta_{nx^i}\|_{\mathbb{L}_{2,\theta}(\mathcal{O},T)} \leq N\|u_x\zeta_{nx}\|_{\mathbb{L}_{2,\theta}(\mathcal{O},T)} \tag{4.15}$$

and, by Lemma 4.5, for each $t \leq T$ we have

$$\|\bar{v}^k_n\|^2_{\mathbb{H}^1_{2,\theta}(\mathcal{O},t)} \leq Nt\|\bar{v}^k_n\|^2_{\mathfrak{H}^2_{2,\theta}(\mathcal{O},t)} \leq Nt\|u_x\zeta_{nx}\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},t)}, \tag{4.16}$$

where N is independent of T since we assume $T \leq 1$. Now, we denote $\bar{u}^k_n := \bar{v}^k_n\psi\bar{\zeta}_n$ and $\bar{u}_n = (\bar{u}^1, \cdot, \bar{u}^{d_1}_n)$. Then $\bar{u}_n$ satisfies

$$d\bar{u}^k_n = (\Delta\bar{u}^k_n + \hat{f}^k_n - a^{ij}_{kr} u^r_{x^j}\zeta_{nx^i})\,dt, \quad \bar{u}^k_n(0) = 0,$$

where $\hat{f}^k_n = -2\bar{v}^k_{nx^i}(\bar{\zeta}_n\psi)_{x^i} - \bar{v}^k_n\Delta(\bar{\zeta}_n\psi)$. Finally, as we denote $u_n := v_n - \bar{u}_n$, we find that u_n satisfies

$$\begin{aligned} du^k_n = (D_i(a^{ij}_{kr}u^r_{nx^j} + \bar{b}^i u_n + \bar{F}^{ik}_n) + b^i_{kr}u^r_{nx^i} + c_{kr}u^r_n + F^k_n)\,dt \\ + (\sigma^i_{kr,m}u^r_{nx^i} + \nu_{kr,m}u^r_n + G^k_{n,m})\,dw^m_t, \end{aligned} \tag{4.17}$$

where

$$\bar{F}^{ik}_n = \bar{f}^{ik}_n + (a^{ij}_{kr} - \delta^{ij}\delta^{kr})\bar{u}^r_{nx^j} + \bar{b}^i_{kr}\bar{u}^r_n,$$

$$F^k_n = f^k_n + \hat{f}^k_n + b^i_{kr}\bar{u}^r_{nx^i} + c_{kr}\bar{u}^r_n, \quad G^k_n = \sigma^i_{kr}\bar{u}^r_{nx^i} + \nu_{kr}\bar{u}^r_n + g^k_n.$$

Then, by Lemmas 4.9, for any $n \geq 1$ and $t \leq T$ we have

$$\|\psi^{-1}u_n\|^2_{\mathbb{H}^1_{2,\theta}(\mathcal{O},t)} \leq N\|\bar{F}^i_n\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},t)} + N\|\psi F_n\|^2_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},t)} + N\|G_n\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},t)}. \tag{4.18}$$

Also, since $u\zeta_0$ has compact support in $\mathcal{O}$, (4.18) holds for $n = 0$ by Theorem 2.4. As we recall that $\psi b, \psi\bar{b}, \psi^2 c, \psi_x, \psi\psi_{xx}, (\bar{\zeta}_n\psi)_x, \psi\Delta(\bar{\zeta}_n\psi)$ are bounded, $\|\cdot\|_{H^{-1}_{2,\theta}(\mathcal{O})} \leq \|\cdot\|_{L_{2,\theta}(\mathcal{O})}$, and

$$\psi^{-1}\bar{u}_n = \bar{\zeta}_n\bar{v}_n, \quad \bar{u}_{nx} = \bar{\zeta}_n\psi\bar{v}_{nx} + \bar{v}_n(\bar{\zeta}_n\psi)_x,$$

we get

$$
\begin{aligned}
& \|\psi(\hat{f}^k_n + b^i_{kr}\bar{u}^r_{nx^i} + c_{kr}\bar{u}^r_n)\|_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},t)} \\
\le\ & N\left(\|\psi\bar{v}_{nx}\|_{\mathbb{L}_{2,\theta}(\mathcal{O},t)} + \|\bar{v}_n\|_{\mathbb{L}_{2,\theta}(\mathcal{O},t)} + \|\bar{u}_{nx}\|_{\mathbb{L}_{2,\theta}(\mathcal{O},t)} + \|\psi^{-1}\bar{u}_n\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},t)}\right) \\
\le\ & N\left(\|\psi\bar{v}_{nx}\|_{\mathbb{L}_{2,\theta}(\mathcal{O},t)} + \|\bar{v}_n\|_{\mathbb{L}_{2,\theta}(\mathcal{O},t)}\right) \\
\le\ & N\|\bar{v}_n\|_{\mathbb{H}^1_{2,\theta}(\mathcal{O},t)}
\end{aligned}
$$

and it leads to

$$
\|\psi F_n\|^2_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},t)} \le N\|\psi f_n\|^2_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},t)} + N\|\bar{v}_n\|^2_{\mathbb{H}^1_{2,\theta}(\mathcal{O},t)}.
$$

Also, by (4.16) we have

$$
\|\bar{v}_n\|^2_{\mathbb{H}^1_{2,\theta}(\mathcal{O},t)} \le Nt\|u_x\zeta_{nx}\|_{\mathbb{L}_{2,\theta}(\mathcal{O},t)}
$$

and consequently

$$
\begin{aligned}
& \sum_n \|\psi F_n\|^2_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},t)} \\
\le\ & N\sum_n\left(\|\psi f_n\|^2_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},t)} + t\|u_x\zeta_{nx}\|^2_{\mathbb{L}_{p,\theta}(\mathcal{O},t)}\right) \\
\le\ & N\|u\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},t)} + Nt\|u_x\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},t)} + N\|\bar{f}\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},t)} + N\|\psi f\|^2_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},t)}.
\end{aligned}
$$

The sums

$$
\sum_n \|\bar{F}^i_n\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},t)}, \quad \sum_n \|G_n\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},t)}.
$$

are estimated similarly. Then for each $t \le T$ one gets

$$
\begin{aligned}
\|\psi^{-1}u\|^2_{\mathbb{H}^1_{2,\theta}(\mathcal{O},t)} &\le N\sum_n \|\psi^{-1}v_n\|^2_{\mathbb{H}^1_{p,\theta}(\mathcal{O},t)} \\
&\le N\|\bar{f}\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},T)} + N\|\psi f\|^2_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},T)} + N\|g\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},T)} \\
&\quad +N\|u\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},t)} + N\cdot t\|\psi^{-1}u\|^2_{\mathbb{H}^1_{2,\theta}(\mathcal{O},t)}.
\end{aligned}
$$

Now, we choose $\varepsilon \in (0,1]$ such that for $t \le T \le \varepsilon$

$$
N\cdot t\|u_x\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},t)} \le 1/2\|\psi^{-1}u\|^2_{\mathbb{H}^1_{2,\theta}(\mathcal{O},t)}.
$$

Then, by Lemma 4.5, for each $t \le T$ we obtain

$$
\begin{aligned}
\|u\|^2_{\mathfrak{H}^1_{2,\theta}(\mathcal{O},t)} &\le N\int_0^t \|u\|^2_{\mathfrak{H}^1_{2,\theta}(\mathcal{O},s)}\,ds + N\|\bar{f}\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},T)} \\
&\quad +N\|\psi f\|^2_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},T)} + N\|g\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},T)}.
\end{aligned} \tag{4.19}
$$

This and Gronwall's inequality lead to the a priori estimate for $T \le \varepsilon$. □

For the case $T \ge \varepsilon$ we need the following lemma, which is proved in [5] for $d_1 = 1$.

Lemma 4.12. *Let $d-1<\theta<d+1+p$, $t_0\leq T$, and $u\in\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},t_0)$ satisfy*

$$du^k(t)=f^k(t)dt+g^k_m(t)dw^m_t,\quad u(0)=0.$$

Then there exists a unique $\tilde{u}\in\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)$ such that $\tilde{u}(t)=u(t)$ for $t\leq t_0(a.s)$ and on $(0,T)$

$$d\tilde{u}^k=(\Delta\tilde{u}^k(t)+\tilde{f}^k(t))dt+g^kI_{t\leq t_0}dw^m_t, \tag{4.20}$$

where $\tilde{f}=(f^k(t)-\Delta u^k(t))I_{t\leq t_0}$. Furthermore, we have

$$\|\tilde{u}\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)}\leq N\|u\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},t_0)}, \tag{4.21}$$

where N is independent of u and t_0.

Proof. We note that for each k, $\tilde{f}^k\in\psi^{-1}\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},T)$ and $g^kI_{t\leq t_0}\in\mathbb{H}^{\gamma+1}_{p,\theta}(\mathcal{O},T)$. Thus, by Theorem 2.9 in [4], equation (4.20) has a unique (real-valued) solution $\tilde{u}^k\in\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)$ and we have

$$\|\tilde{u}^k\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},T)}\leq N\|u^k\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(\mathcal{O},t_0)}. \tag{4.22}$$

We define $\tilde{u}=(\tilde{u}^1,\tilde{u}^2,\cdots,\tilde{u}^{d_1})$. To show $\tilde{u}(t)=u(t)$ for $t\leq t_0$ we notice that, for $t\leq t_0$, the function $v^k(t)=\tilde{u}^k(t)-u^k(t)$ satisfies the equation

$$dv^k(t)=\Delta v^k\,dt,\quad v(0,\cdot)=0.$$

Thus, by Theorem 2.9 in [4], $v^k(t)=0$ for $t\leq t_0\,(a.e)$. The lemma is proved. □

We finish the proof of Theorem 4.7.

Proof of Theorem 4.7 As usual, we only prove that estimate (4.7) holds given that a solution u already exists. For simplicity, we assume $u_0=0$. See the proof of Theorem 5.1 in [10] for the general case.

Take an integer $M\geq 2$ such that $T/M\leq\varepsilon$ and we denote $t_n=Tn/M$. Assume that, for $n=1,2,...,M-1$, we have the estimate (4.7) with t_n in place of T (and N depending only on d,d_1,θ,δ,K,L and T). We use the induction on n.

Let $u_n\in\mathfrak{H}^1_{2,\theta}$ be the continuation of u on $[t_n,T]$, which exists by Lemma 4.12 with $\gamma=-1$ and $t_0=t_n$. As we denote $v_n:=u-u_n$, we have $v_n(t)=0$ for $t\leq t_n(a.s)$ and, for any $t\in[t_n,T]$ and $\phi\in C^\infty_0(\mathcal{O})$,

$$\begin{aligned}(v^k_n(t),\phi)\quad=\quad&-\int_{t_n}^t(a^{ij}_{kr}v^r_{nx^j}+\bar{b}^i_{kr}v^r_n+\bar{f}^{ik}_n,\phi_{x^i})(s)ds+\int_{t_n}^t(b^i_{kr}v^r_{nx^i}+c_{kr}v^r_n+f_n,\phi)(s)ds\\&+\int_{t_n}^t(\sigma^i_{kr,m}u^r_{nx^i}+\nu_{kr,m}v^r_n+g^k_{n,m},\phi)(s)dw^m_s,\end{aligned}$$

where

$$\bar{f}^{ik}_n:=(a^{ij}_{kr}-\delta^{ij}\delta^{kr})u^r_{nx^j}+\bar{b}^i_{kr}u^r_n+\bar{f}^{ik},\quad f^k_n=b^i_{kr}u^r_{nx^i}+c_{kr}u^r_m+f^k,$$

$$g^k_n:=\sigma^i_{kr}u^r_{nx^i}+\nu_{kr}u^r_n+g^k.$$

Next, instead of random processes on $[0,T]$ we consider processes given on $[t_n,T]$ and introduces spaces $\mathfrak{H}^{\gamma}_{p,\theta}(\mathcal{O},[t_n,T])$, $\mathbb{L}_{p,\theta}(\mathcal{O},[t_n,t])$, $\mathbb{H}^{\gamma}_{p,\theta}(\mathcal{O},[t_n,T])$ in a natural way. Then we get a counterpart of the previous result and conclude that

$$\begin{aligned} & \mathbb{E}\int_{t_n}^{t_{n+1}} \|\psi^{-1}(u-u_n)(s)\|^2_{H^1_{2,\theta}(\mathcal{O})}ds \\ \leq \quad & N\mathbb{E}\int_{t_n}^{t_{n+1}} (\|\bar{f}^i_n(s)\|^2_{L_{2,\theta}(\mathcal{O})} + \|\psi f_n(s)\|^2_{H^{-1}_{2,\theta}(\mathcal{O})} + \|g_n(s)\|^2_{L_{2,\theta}(\mathcal{O})})ds. \end{aligned}$$

Thus, by the induction hypothesis we get

$$\begin{aligned} & \mathbb{E}\int_0^{t_{n+1}} \|\psi^{-1}u(s)\|^2_{H^1_{2,\theta}(\mathcal{O})}ds \\ \leq \quad & N\mathbb{E}\int_0^T \|\psi^{-1}u_n(s)\|^2_{H^1_{2,\theta}(\mathcal{O})}ds + N\mathbb{E}\int_{t_n}^{t_{n+1}} \|\psi^{-1}(u-u_n)(s)\|^2_{H^1_{2,\theta}(\mathcal{O})}ds \\ \leq \quad & N(\|\bar{f}^i\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},t_{n+1})} + \|\psi f\|^2_{\mathbb{H}^{-1}_{2,\theta}(\mathcal{O},t_{n+1})} + \|g\|^2_{\mathbb{L}_{2,\theta}(\mathcal{O},t_{n+1},\ell_2)}). \end{aligned}$$

We see that the induction goes through and thus the theorem is proved.

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